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Low-frequency scattering by passive periodic structures for oblique incidence: low-pass case

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Abstract

We derive the low-frequency behavior of the scattering coefficients from a low-pass structure which is periodic in a plane, and finite in the normal direction. The analysis is for oblique incidence of arbitrary polarization on a structure which can be anisotropic in both electric and magnetic material properties, and may contain metal inclusions. The metal inclusions can be modeled both as perfect electric conductors (PEC) and with a finite conductivity. It is found that the low-frequency reflection and transmission coefficients are proportional to the sum and difference of the electric and magnetic polarizabilities per unit area of the periodic structure. If the metal inclusions are modeled as PEC instead of as a finite conductivity, the first-order low-frequency reflection is larger, whereas the first-order transmission is smaller.

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1. Introduction

Periodic structures are often used as spatial filters, or frequency selective surfaces. They are typically either band pass or band stop. Band-pass structures usually consist of one or several metal sheets with periodic arrays of apertures, whereas the band-stop structures are usually periodic arrays of metal inclusions; in the case of one sheet, the two concepts can be considered as complementary structures via Babinet's principle.

In a series of papers, physical limitations on the amount of electromagnetic interaction available for antennas, materials and general scatterers have been derived based solely on the principles of linearity, causality and energy conservation [1–5]. There, it is demonstrated that the low-frequency behavior of the structure under consideration provides a measure of the total electromagnetic interaction available for all frequencies, much in the same spirit as classical sum rules [6]. It is anticipated that the same kind of relations can be derived for many kinds of periodic structures as well, which makes it interesting to take an explicit

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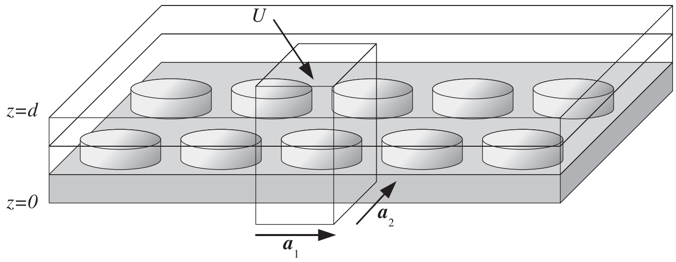

Figure 1. Typical geometry of the periodic structure.

look at low-frequency scattering for such structures. Some published results already exist for electromagnetic absorbers consisting of a layered structure on a metal sheet [7], for artificial magnetic ground planes [8] and for transmission at normal incidence through band-stop structures [9]. The latter case is a direct application of the results in this paper, and shows that the amount of electromagnetic power that can be blocked from transmission through a band-stop screen is bounded by the static polarizability per unit area of the screen.

Passive periodic structures have been studied extensively. A full review is beyond the scope of this paper, but we point out that many important low-frequency results for specific geometries (typically metallic inclusions in vacuum) are available in the literature, see for instance [10–12] and references therein. When the structures are embedded in a complex material, possibly layered, more advanced Green's functions can be used [13, 14] to extend the results, at least in principle.

In this paper, we use an explicit analytical approach to identify the electric and magnetic polarizabilities per unit area as the quantities of interest in section 3, and state the equations which need to be solved in order to compute them in section 4. We limit ourselves to the band-stop case, since this is the one most easily analyzed. The reason for this is that in the static limit of band-stop structures, the tangential electric and magnetic fields are continuous. In the band-pass case, the possibility of an interelement current in the metal sheets provides a possibility for discontinuous tangential magnetic fields, which must be handled separately.

Our analysis is for oblique incidence with arbitrary polarization, and includes fully anisotropic permittivities and permeabilities, as well as metal inclusions. As a result, the first-order asymptotic reflection and transmission coefficients can be computed if the electric and magnetic static polarizabilities per unit area are known. Since these polarizabilities are calculated from a static problem, there exist variational principles that make it possible to give bounds for the polarizabilities even for very complicated structures.

2. Notation

Let the periodic structure be situated between $0 < z < d$, with periodicity described by two basis vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the xy plane as in figure 1. These are the sides of the unit cell U with area $A = \hat{\boldsymbol{z}} \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)$. An arbitrary lattice vector is then described by

$$\boldsymbol{x}_{\boldsymbol{n}} = n_1 \boldsymbol{a}_1 + n_2 \boldsymbol{a}_2, \tag{1}$$

with n_1 and n_2 being integers. The material parameters are U-periodic, i.e. $\boldsymbol{\epsilon}(\boldsymbol{x} + \boldsymbol{x}_{\boldsymbol{n}}) = \boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x} + \boldsymbol{x}_{\boldsymbol{n}}) = \boldsymbol{\mu}(\boldsymbol{x})$ for all $\boldsymbol{n} = (n_1, n_2)$, where $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the permittivity and permeability matrices, respectively. In the regions $z < 0$ and $z > d$ we have $\boldsymbol{\epsilon}(\boldsymbol{x}) = \epsilon_0 \mathbf{I}$ and

$\boldsymbol{\mu}(\boldsymbol{x}) = \mu_0\mathbf{I}$, where ϵ_0 and μ_0 are the permittivity and permeability of vacuum, respectively. Let the incident field be a plane wave (time convention $\mathrm{e}^{-\mathrm{i}\omega t}$)

$$\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x}) = \boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2}$$

where the constant vector $\boldsymbol{E}_0$ is the polarization, and $\boldsymbol{k}$ is the wave vector of the incident wave. The amplitude of the wave vector is $|\boldsymbol{k}| = k = \omega/c$, where c is the speed of light in vacuum. The wave vector can be separated in one normal and one transverse part,

$$\boldsymbol{k} = \boldsymbol{k}_\perp + k_z\hat{\boldsymbol{z}}, \qquad \frac{|\boldsymbol{k}_\perp|}{k} = \sin\theta, \qquad \frac{k_z}{k} = \cos\theta, \tag{3}$$

where θ is the angle of incidence and $\boldsymbol{k}_\perp$ is a vector in the xy plane.

In the surrounding air, the transverse components of the electric and magnetic fields in a propagating plane wave are related by an impedance matrix $\mathbf{Z}$:

$$\boldsymbol{E}_\perp = \pm\mathbf{Z}(-\hat{\boldsymbol{z}}\times\boldsymbol{H}_\perp) \quad \Leftrightarrow \quad \boldsymbol{H}_\perp = \pm\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_\perp, \tag{4}$$

where the upper sign is for waves propagating in the positive z-direction, and the lower sign is for waves propagating in the negative z-direction. The impedance matrix $\mathbf{Z}$ has eigenvalues $\eta_0\cos\theta$ (TM case) and $\eta_0/\cos\theta$ (TE case), where $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 377\Omega$ is the intrinsic wave impedance in vacuum. When $\boldsymbol{k}_\perp \neq \mathbf{0}$, it can be given the explicit representation (where $\boldsymbol{k}'_\perp = \hat{\boldsymbol{z}}\times\boldsymbol{k}_\perp$ is orthogonal to $\boldsymbol{k}_\perp$)

$$\mathbf{Z} = \eta_0\cos\theta\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2} + \frac{\eta_0}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2}, \tag{5}$$

and for $\boldsymbol{k}_\perp = \mathbf{0}$, corresponding to $\cos\theta = 1$, we have $\mathbf{Z} = \eta_0\mathbf{I}$. We use here a dyadic notation, where the action of the impedance matrix on an arbitrary transverse vector $\boldsymbol{A}$ is understood as $\mathbf{Z}\boldsymbol{A} = \eta_0\cos\theta\frac{\boldsymbol{k}_\perp(\boldsymbol{k}_\perp\cdot\boldsymbol{A})}{|\boldsymbol{k}_\perp|^2} + \frac{\eta_0}{\cos\theta}\frac{\boldsymbol{k}'_\perp(\boldsymbol{k}'_\perp\cdot\boldsymbol{A})}{|\boldsymbol{k}'_\perp|^2}$.

Due to the use of a plane wave as excitation and the periodicity of the structure, the fields (including incident and scattered fields) satisfy the following translation property:

$$\boldsymbol{E}(\boldsymbol{x}+\boldsymbol{x}_n) = \boldsymbol{E}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}_n}, \tag{6}$$

where $\boldsymbol{x}_n$ is an arbitrary lattice vector. This property implies that the field

$$\widetilde{\boldsymbol{E}}(\boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}\boldsymbol{E}(\boldsymbol{x}) \tag{7}$$

is U-periodic in $\boldsymbol{x}$. The periodic field $\widetilde{\boldsymbol{E}}(\boldsymbol{x})$ is called the Bloch amplitude of the field $\boldsymbol{E}(\boldsymbol{x})$ [15, 16].

3. Low-frequency behavior

Maxwell's equations for time-harmonic fields are (where the possibly anisotropic matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the permittivity and permeability of the material, respectively)

$$\nabla\times\boldsymbol{E} = \mathrm{i}\omega\boldsymbol{\mu}(\boldsymbol{x})\boldsymbol{H} \tag{8}$$

$$\nabla\times\boldsymbol{H} = -\mathrm{i}\omega\boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{E}. \tag{9}$$

Multiplying these fields with the transverse phase factor of the incident field, $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}$, we obtain the equations for the Bloch amplitudes (using the identity $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}\nabla\times\boldsymbol{E} = (\nabla+\mathrm{i}\boldsymbol{k}_\perp)\times\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}\boldsymbol{E}$)

$$(\nabla+\mathrm{i}\boldsymbol{k}_\perp)\times\widetilde{\boldsymbol{E}} = \mathrm{i}\omega\boldsymbol{\mu}\widetilde{\boldsymbol{H}} \tag{10}$$

$$(\nabla+\mathrm{i}\boldsymbol{k}_\perp)\times\widetilde{\boldsymbol{H}} = -\mathrm{i}\omega\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}. \tag{11}$$

Integrating over $(x, y) \in U$ and $z_1 < z < z_2$, where $z_1 < 0$ and $z_2 > d$ are chosen so that the structure is enclosed, implies

$$\hat{\boldsymbol{z}} \times \left(\int_U \widetilde{\boldsymbol{E}}(z_2)\,\mathrm{d}S - \int_U \widetilde{\boldsymbol{E}}(z_1)\,\mathrm{d}S \right) = \int_{z_1}^{z_2} \int_U (-\mathrm{i}\boldsymbol{k}_\perp \times \widetilde{\boldsymbol{E}} + \mathrm{i}\omega\boldsymbol{\mu}\widetilde{\boldsymbol{H}})\,\mathrm{d}S\,\mathrm{d}z \tag{12}$$

$$\hat{\boldsymbol{z}} \times \left(\int_U \widetilde{\boldsymbol{H}}(z_2)\,\mathrm{d}S - \int_U \widetilde{\boldsymbol{H}}(z_1)\,\mathrm{d}S \right) = \int_{z_1}^{z_2} \int_U \left(-\mathrm{i}\boldsymbol{k}_\perp \times \widetilde{\boldsymbol{H}} - \mathrm{i}\omega\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}} \right) \mathrm{d}S\,\mathrm{d}z. \tag{13}$$

We use the following notation for the mean value of the fields (where $h = z_2 - z_1$)

$$\bar{\boldsymbol{E}} = \frac{1}{Ah} \int_{z_1}^{z_2} \int_U \widetilde{\boldsymbol{E}}\,\mathrm{d}S\,\mathrm{d}z \qquad \bar{\boldsymbol{E}}_{1,2} = \frac{1}{A} \int_U \widetilde{\boldsymbol{E}}(z_{1,2})\,\mathrm{d}S \tag{14}$$

$$\bar{\boldsymbol{H}} = \frac{1}{Ah} \int_{z_1}^{z_2} \int_U \widetilde{\boldsymbol{H}}\,\mathrm{d}S\,\mathrm{d}z \qquad \bar{\boldsymbol{H}}_{1,2} = \frac{1}{A} \int_U \widetilde{\boldsymbol{H}}(z_{1,2})\,\mathrm{d}S. \tag{15}$$

The following matrices γ_e and γ_m exist and are bounded as $\omega \to 0$, since they represent the response of a linear system on an excitation $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ (see section 4 for computing the matrices in the static limit and generalization to the case of metallic inclusions):

$$\int_{z_1}^{z_2} \int_U (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})\widetilde{\boldsymbol{E}}\,\mathrm{d}S\,\mathrm{d}z \overset{\mathrm{def}}{=} \gamma_\mathrm{e}\boldsymbol{E}_0 \tag{16}$$

$$\int_{z_1}^{z_2} \int_U (\boldsymbol{\mu}/\mu_0 - \mathbf{I})\widetilde{\boldsymbol{H}}\,\mathrm{d}S\,\mathrm{d}z \overset{\mathrm{def}}{=} \gamma_\mathrm{m}\boldsymbol{H}_0. \tag{17}$$

The equations are then

$$\hat{\boldsymbol{z}} \times (\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1) = -\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{E}} + \mathrm{i}\omega\mu_0 h\bar{\boldsymbol{H}} + \mathrm{i}\omega\mu_0 A^{-1}\gamma_\mathrm{m}\boldsymbol{H}_0 \tag{18}$$

$$\hat{\boldsymbol{z}} \times (\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1) = -\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{H}} - \mathrm{i}\omega\epsilon_0 h\bar{\boldsymbol{E}} - \mathrm{i}\omega\epsilon_0 A^{-1}\gamma_\mathrm{e}\boldsymbol{E}_0, \tag{19}$$

where the factor $\boldsymbol{k}_\perp h$ is dimensionless, the factors $\mathrm{i}\omega\mu_0 h$ and $\mathrm{i}\omega\mu_0 A^{-1}\gamma_\mathrm{m}$ have dimensions of impedance, and the factors $\mathrm{i}\omega\epsilon_0 h$ and $\mathrm{i}\omega\epsilon_0 A^{-1}\gamma_\mathrm{e}$ have dimensions of admittance.

Since the right-hand side of each of these equations is proportional to ω, this shows us that in the static limit we have

$$\lim_{\omega\to 0} \hat{\boldsymbol{z}} \times (\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1) = \mathbf{0} \qquad \text{and} \qquad \lim_{\omega\to 0} \hat{\boldsymbol{z}} \times (\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1) = \mathbf{0}, \tag{20}$$

i.e. the static tangential fields are continuous across the structure. It is tempting to consider the first two terms on the right-hand sides of (18) and (19) as due only to propagation in vacuum, but as we show in the following they also contain a term contributing to the transmission and reflection coefficients when considering oblique incidence.

3.1. Rewriting in transverse components

Since the left-hand sides of our equations are orthogonal to $\hat{\boldsymbol{z}}$ due to the cross product with $\hat{\boldsymbol{z}}$, the z-component of the right-hand sides must be zero,

$$0 = -\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{E}}_\perp) + \mathrm{i}\omega\mu_0 h\bar{H}_z + \mathrm{i}\omega\mu_0 A^{-1}(\gamma_\mathrm{m}\boldsymbol{H}_0)_z \tag{21}$$

$$0 = -\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{H}}_\perp) - \mathrm{i}\omega\epsilon_0 h\bar{E}_z - \mathrm{i}\omega\epsilon_0 A^{-1}(\gamma_\mathrm{e}\boldsymbol{E}_0)_z, \tag{22}$$

From this we extract the z components as

$$\bar{E}_z = \frac{-\hat{\boldsymbol{z}}\cdot(\mathrm{i}\boldsymbol{k}_\perp \times \bar{\boldsymbol{H}}_\perp)}{\mathrm{i}\omega\epsilon_0} - \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{Ah} = \frac{-\mathrm{i}\boldsymbol{k}'_\perp\cdot\bar{\boldsymbol{H}}_\perp}{\mathrm{i}\omega\epsilon_0} - \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{Ah} \tag{23}$$

$$\bar{H}_z = \frac{\hat{\boldsymbol{z}}\cdot(\mathrm{i}\boldsymbol{k}_\perp \times \bar{\boldsymbol{E}}_\perp)}{\mathrm{i}\omega\mu_0} - \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{Ah} = \frac{\mathrm{i}\boldsymbol{k}'_\perp\cdot\bar{\boldsymbol{E}}_\perp}{\mathrm{i}\omega\mu_0} - \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{Ah}, \tag{24}$$

where we used $\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}_\perp\times\bar{\boldsymbol{E}}_\perp) = (\hat{\boldsymbol{z}}\times\boldsymbol{k}_\perp)\cdot\bar{\boldsymbol{E}}_\perp = \boldsymbol{k}'_\perp\cdot\bar{\boldsymbol{E}}_\perp$. Inserting this into the transverse part of the equations implies

$$\begin{aligned}\hat{\boldsymbol{z}}\times(\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1) &= \underbrace{-\mathrm{i}\boldsymbol{k}_\perp h \times \hat{\boldsymbol{z}}\bar{E}_z}_{=\mathrm{i}\boldsymbol{k}'_\perp h\bar{E}_z} + \mathrm{i}\omega\mu_0 h\bar{\boldsymbol{H}}_\perp + \mathrm{i}\omega\mu_0 A^{-1}(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp \\ &= \frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{\mathrm{i}\omega\epsilon_0}h\cdot\bar{\boldsymbol{H}}_\perp - \mathrm{i}\boldsymbol{k}'_\perp\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} + \mathrm{i}\omega\mu_0 h\bar{\boldsymbol{H}}_\perp + \mathrm{i}\omega\mu_0\frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A}\end{aligned} \tag{25}$$

and

$$\begin{aligned}\hat{\boldsymbol{z}}\times(\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1) &= \underbrace{-\mathrm{i}\boldsymbol{k}_\perp h \times \hat{\boldsymbol{z}}\bar{H}_z}_{=\mathrm{i}\boldsymbol{k}'_\perp h\bar{H}_z} - \mathrm{i}\omega\epsilon_0 h\bar{\boldsymbol{E}}_\perp - \mathrm{i}\omega\epsilon_0 A^{-1}(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp \\ &= -\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{\mathrm{i}\omega\mu_0}h\cdot\bar{\boldsymbol{E}}_\perp - \mathrm{i}\boldsymbol{k}'_\perp\frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} - \mathrm{i}\omega\epsilon_0 h\bar{\boldsymbol{E}}_\perp - \mathrm{i}\omega\epsilon_0\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A}.\end{aligned} \tag{26}$$

The terms proportional to $(\gamma_\mathrm{e}\boldsymbol{E}_0)_z$ and $(\gamma_\mathrm{m}\boldsymbol{H}_0)_z$ are the extra contributions due to oblique incidence.

3.2. Reflection and transmission

Let $\mathbf{t}$ and $\mathbf{r}$ denote the transmission and reflection matrix, respectively, for the transverse electric field with the reference plane $z = 0$, i.e. $\boldsymbol{E}^\mathrm{t}_\perp(z) = \mathrm{e}^{\mathrm{i}k_z z}\mathbf{t}\boldsymbol{E}^\mathrm{i}_\perp(0)$ and $\boldsymbol{E}^\mathrm{r}_\perp(z) = \mathrm{e}^{-\mathrm{i}k_z z}\mathbf{r}\boldsymbol{E}^\mathrm{i}_\perp(0)$. When considering the low frequency limit, the reflection and transmission matrices are expanded in a formal power series in ω as

$$\mathbf{r}(\omega) = \mathbf{r}_0 + \omega\mathbf{r}_1 + \cdots \tag{27}$$

$$\mathbf{t}(\omega) = \mathbf{t}_0 + \omega\mathbf{t}_1 + \cdots. \tag{28}$$

Since the static tangential fields are continuous across the structure according to (20), it is immediately seen that $\mathbf{r}_0 = \mathbf{0}$ and $\mathbf{t}_0 = \mathbf{I}$, as expected for a low-pass structure. In this paper, we are only interested in terms up to $\mathbf{r}_1$ and $\mathbf{t}_1$, which means it is sufficient to keep only terms up to first order in $\bar{\boldsymbol{E}}_{1,2}$ and $\bar{\boldsymbol{H}}_{1,2}$ (we use (4) to represent the transverse magnetic fields using $\boldsymbol{E}_{0\perp}$ and suppress the expansion of $\mathbf{r}$ and $\mathbf{t}$ for brevity):

$$\bar{\boldsymbol{E}}_{1\perp} = (\mathbf{I} + \mathbf{r} + \mathrm{i}k_z z_1\mathbf{I})\boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_{1\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}(\mathbf{I} - \mathbf{r} + \mathrm{i}k_z z_1\mathbf{I})\boldsymbol{E}_{0\perp} \tag{29}$$

$$\bar{\boldsymbol{E}}_{2\perp} = (\mathbf{t} + \mathrm{i}k_z z_2\mathbf{I})\boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_{2\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}(\mathbf{t} + \mathrm{i}k_z z_2\mathbf{I})\boldsymbol{E}_{0\perp} \tag{30}$$

$$\bar{\boldsymbol{E}}_\perp = \boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_\perp = \boldsymbol{H}_{0\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{31}$$

The fields $\bar{\boldsymbol{E}}_\perp$ and $\bar{\boldsymbol{H}}_\perp$ are expanded only to zeroth order, since in the equations they are multiplied by factors proportional to ω. In order for $\bar{\boldsymbol{E}}_\perp = \boldsymbol{E}_{0\perp}$ and $\bar{\boldsymbol{H}}_\perp = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_{0\perp}$ to hold to zeroth order, we need to consider a limit process where $h \to \infty$ and $kh \to 0$ simultaneously. This may seem to invalidate expansions (27) and (28) since an extra scale is introduced, but a deeper analysis shows the expansions are still valid.

The incident field satisfies (set all polarizability matrices in (25) and (26) to zero and use $\boldsymbol{E}_1 = \boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}k_z z_1}$, $\boldsymbol{E}_2 = \boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}k_z z_2}$, etc, and expand to the first order)

$$(\mathrm{e}^{\mathrm{i}k_z z_2} - \mathrm{e}^{\mathrm{i}k_z z_1})\hat{\boldsymbol{z}} \times \boldsymbol{E}_0 = \mathrm{i}k_z(z_2 - z_1)\hat{\boldsymbol{z}} \times \boldsymbol{E}_{0\perp} = \left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\epsilon_0}h + \mathrm{i}\omega\mu_0 h\mathbf{I}\right]\boldsymbol{H}_{0\perp} \tag{32}$$

$$(\mathrm{e}^{\mathrm{i}k_z z_2} - \mathrm{e}^{\mathrm{i}k_z z_1})\hat{\boldsymbol{z}} \times \boldsymbol{H}_0 = \mathrm{i}k_z(z_2 - z_1)\hat{\boldsymbol{z}} \times \boldsymbol{H}_{0\perp} = -\left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\mu_0}h + \mathrm{i}\omega\epsilon_0 h\mathbf{I}\right]\boldsymbol{E}_{0\perp}. \tag{33}$$

Subtracting this result from (25) and (26), and using expansions (29)–(31), we find (after multiplying (25) by $-\hat{\boldsymbol{z}}\times$ and (26) by $-\mathbf{Z}$, and observing that $-\hat{\boldsymbol{z}}\times\boldsymbol{k}'_\perp = -\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{k}_\perp) = \boldsymbol{k}_\perp$, as well as the relations $k = \omega\sqrt{\epsilon_0\mu_0}$ and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$)

$$(\mathbf{t} - \mathbf{I} - \mathbf{r})\boldsymbol{E}_{0\perp} = -\mathrm{i}\boldsymbol{k}_\perp \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} - \mathrm{i}k\eta_0\hat{\boldsymbol{z}} \times \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} \tag{34}$$

$$(\mathbf{t} - \mathbf{I} + \mathbf{r})\boldsymbol{E}_{0\perp} = \mathrm{i}\mathbf{Z}\boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} + \mathrm{i}k\eta_0^{-1}\mathbf{Z}\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A}. \tag{35}$$

We recall the fact that this equation is only valid asymptotically to the first order as $k \to 0$.

3.3. *Solving for the reflection and transmission matrices*

To find explicit expressions for the transmission and reflection matrices, we must express all the field components on the right-hand sides in $\boldsymbol{E}_{0\perp}$, so that this factor can be eliminated. This can be done from the knowledge that the incident field is a plane wave in the surrounding medium, which implies the following formulas:

$$E_{0z} = -\eta_0 \frac{\boldsymbol{k}_\perp \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp}}{k} \tag{36}$$

$$\boldsymbol{H}_{0\perp} = \hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{37}$$

$$H_{0z} = \eta_0^{-1}\frac{\boldsymbol{k}'_\perp \cdot \boldsymbol{E}_{0\perp}}{k}. \tag{38}$$

Making use of these representations and the decompositions

$$\gamma_\mathrm{e} = \begin{pmatrix} \gamma_{\mathrm{e}\perp\perp} & \gamma_{\mathrm{e}\perp z} \\ \gamma_{\mathrm{e}z\perp} & \gamma_{\mathrm{e}zz} \end{pmatrix}, \qquad \text{and} \qquad \gamma_\mathrm{m} = \begin{pmatrix} \gamma_{\mathrm{m}\perp\perp} & \gamma_{\mathrm{m}\perp z} \\ \gamma_{\mathrm{m}z\perp} & \gamma_{\mathrm{m}zz} \end{pmatrix}, \tag{39}$$

we can write the various components of the dipole moments as

$$(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp = \gamma_{\mathrm{e}\perp\perp}\boldsymbol{E}_{0\perp} - \eta_0\gamma_{\mathrm{e}\perp z}\frac{\boldsymbol{k}_\perp}{k} \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{40}$$

$$(\gamma_\mathrm{e}\boldsymbol{E}_0)_z = \gamma_{\mathrm{e}z\perp}\boldsymbol{E}_{0\perp} - \eta_0\gamma_{\mathrm{e}zz}\frac{\boldsymbol{k}_\perp}{k} \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{41}$$

$$(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp = \gamma_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} + \eta_0^{-1}\gamma_{\mathrm{m}\perp z}\frac{\boldsymbol{k}'_\perp}{k} \cdot \boldsymbol{E}_{0\perp} \tag{42}$$

$$(\gamma_\mathrm{m}\boldsymbol{H}_0)_z = \gamma_{\mathrm{m}z\perp}\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} + \eta_0^{-1}\gamma_{\mathrm{m}zz}\frac{\boldsymbol{k}'_\perp}{k} \cdot \boldsymbol{E}_{0\perp}. \tag{43}$$

Collecting all the results, we can write the transmission and reflection matrices as

$$\mathbf{t}-\mathbf{I} = \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\left[\frac{\gamma_{\mathrm{e}\perp\perp}}{A}+\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{k^2}\frac{\gamma_{\mathrm{m}zz}}{A}\right]+\left[-\hat{\boldsymbol{z}}\times\frac{\gamma_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times+\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\right]\mathbf{Z}^{-1}\eta_0 + \mathbf{Z}\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\gamma_{\mathrm{m}z\perp}}{A}-\frac{\gamma_{\mathrm{e}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}+\hat{\boldsymbol{z}}\times\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\gamma_{\mathrm{e}z\perp}}{A}-\frac{\gamma_{\mathrm{m}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\right\} \tag{44}$$

$$\mathbf{r} = \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\left[\frac{\gamma_{\mathrm{e}\perp\perp}}{A}+\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{k^2}\frac{\gamma_{\mathrm{m}zz}}{A}\right]-\left[-\hat{\boldsymbol{z}}\times\frac{\gamma_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times+\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\right]\mathbf{Z}^{-1}\eta_0 + \mathbf{Z}\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\gamma_{\mathrm{m}z\perp}}{A}-\frac{\gamma_{\mathrm{e}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}-\hat{\boldsymbol{z}}\times\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\gamma_{\mathrm{e}z\perp}}{A}-\frac{\gamma_{\mathrm{m}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\right\}. \tag{45}$$

For normal incidence, where $\boldsymbol{k}_\perp = \boldsymbol{k}'_\perp = \mathbf{0}$, the result simplifies to

$$\mathbf{t}-\mathbf{I} = \frac{\mathrm{i}k}{2}\left\{\frac{\gamma_{\mathrm{e}\perp\perp}}{A}-\hat{\boldsymbol{z}}\times\frac{\gamma_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times\right\} \tag{46}$$

$$\mathbf{r} = \frac{\mathrm{i}k}{2}\left\{\frac{\gamma_{\mathrm{e}\perp\perp}}{A}+\hat{\boldsymbol{z}}\times\frac{\gamma_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times\right\}. \tag{47}$$

Note that since the operation $\hat{\boldsymbol{z}}\times$ can be identified with a skew-symmetric matrix which is its own (negative) inverse, the matrix $-\hat{\boldsymbol{z}}\times\gamma_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}}\times = (\hat{\boldsymbol{z}}\times)^{-1}\gamma_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}}\times$ is a similarity transform of $\gamma_{\mathrm{m}\perp\perp}$. This demonstrates that the first-order correction to the static transmission and reflection coefficients is given by the sum and difference of the electric and magnetic polarizabilities per unit area of the structure, multiplied by $\mathrm{i}k/2$. Note that the expressions contain both co- and cross-polarization results.

Example: dielectric film. For a dielectric, nonmagnetic film we can compute both the polarizability matrices and the transmission and reflection coefficients explicitly. The film is contained in the region $0 < z < d$, and has the isotropic permittivity $\boldsymbol{\epsilon} = \epsilon_{\mathrm{r}}\epsilon_0\mathbf{I}$ and vacuum permeability $\boldsymbol{\mu} = \mu_0\mathbf{I}$. The magnetic polarizability is then $\gamma_{\mathrm{m}} = \mathbf{0}$, and the electric polarizability is

$$\gamma_{\mathrm{e}} = \begin{pmatrix}\gamma_1 & 0 & 0\\ 0 & \gamma_1 & 0\\ 0 & 0 & \gamma_2\end{pmatrix}, \qquad \text{where} \qquad \begin{cases}\gamma_1 = (\epsilon_{\mathrm{r}}-1)Ad\\ \gamma_2 = \left(1-\epsilon_{\mathrm{r}}^{-1}\right)Ad.\end{cases} \tag{48}$$

Should the film consist of several layers, the polarizabilities become $\gamma_1 = A\sum_n(\epsilon_{\mathrm{r},n}-1)d_n$ and $\gamma_2 = A\sum_n\left(1-\epsilon_{\mathrm{r},n}^{-1}\right)d_n$, where $\epsilon_{\mathrm{r},n}$ and d_n are the relative permittivity and thickness of layer n, respectively. Expressions (44) and (45) are then

$$\begin{aligned}\mathbf{t}-\mathbf{I} &= \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\frac{\gamma_{\mathrm{e}\perp\perp}}{A}+\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\mathbf{Z}^{-1}\eta_0\right\}\\ &= \frac{\mathrm{i}kd}{2}\left\{\cos\theta\left[\epsilon_{\mathrm{r}}-1+\left(1-\epsilon_{\mathrm{r}}^{-1}\right)\tan^2\theta\right]\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2}+\frac{\epsilon_{\mathrm{r}}-1}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2}\right\}\end{aligned} \tag{49}$$

$$\begin{aligned}\mathbf{r} &= \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\frac{\gamma_{\mathrm{e}\perp\perp}}{A}-\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\mathbf{Z}^{-1}\eta_0\right\}\\ &= \frac{\mathrm{i}kd}{2}\left\{\cos\theta\left[\epsilon_{\mathrm{r}}-1-\left(1-\epsilon_{\mathrm{r}}^{-1}\right)\tan^2\theta\right]\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2}+\frac{\epsilon_{\mathrm{r}}-1}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2}\right\},\end{aligned} \tag{50}$$

which can be confirmed to be the proper low-frequency expansion of the transmission and reflection coefficients for a dielectric film [17, p 65].

4. Polarizability matrix

We now turn to the problem of computing the polarizability matrices γ_e and γ_m in the static limit.

4.1. Finite material parameters with no conductivity

We use Stevenson's method [18] to extract the low-frequency equations, as is traditional in homogenization theory [19]. A formal expansion of the fields in a power series in ω, i.e.

$$\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{E}}^{(0)} + \omega\widetilde{\boldsymbol{E}}^{(1)} + \cdots \tag{51}$$

$$\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{H}}^{(0)} + \omega\widetilde{\boldsymbol{H}}^{(1)} + \cdots . \tag{52}$$

and identifying similar powers of ω in the equations implies that Maxwell's equations reduce to the static equations for the zeroth-order fields (where the material parameters must be understood as the static limit, i.e. $\boldsymbol{\epsilon}(\boldsymbol{x}) = \lim_{\omega\to 0}\boldsymbol{\epsilon}(\boldsymbol{x},\omega)$ and $\boldsymbol{\mu}(\boldsymbol{x}) = \lim_{\omega\to 0}\boldsymbol{\mu}(\boldsymbol{x},\omega)$)

$$\nabla\times\widetilde{\boldsymbol{E}}^{(0)} = \mathbf{0} \qquad \nabla\cdot[\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)}] = 0 \tag{53}$$

$$\nabla\times\widetilde{\boldsymbol{H}}^{(0)} = \mathbf{0} \qquad \nabla\cdot[\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)}] = 0 \tag{54}$$

with periodic boundary conditions in the xy plane. In the z-direction, we require that $\widetilde{\boldsymbol{E}}^{(0)}$ and $\widetilde{\boldsymbol{H}}^{(0)}$ go to constants $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ as $z\to\pm\infty$. The zero-curl condition implies that

$$\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\phi_e \qquad \text{and} \qquad \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{H}_0 - \nabla\phi_m, \tag{55}$$

where the potentials ϕ_e and ϕ_m are U-periodic functions in x and y with zero mean over U, and $\nabla\phi_e$ and $\nabla\phi_m$ both decay to zero as $z\to\pm\infty$ and are square integrable. Note that we do not require ϕ_e and ϕ_m to be zero at infinity. That this cannot be the general case is seen from a dielectric film subjected to a field in the z-direction. The discontinuous polarization in the z-direction induces surface charges on the boundaries of the film, which in turn implies a potential difference between the sides of the film. Thus, the potential cannot in general be zero on both sides.

We can now summarize the low-frequency problem as two separate local problems in the unit cell:

$$\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\phi_e)] = 0 \tag{56}$$

$$\nabla\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0 - \nabla\phi_m)] = 0 \tag{57}$$

for prescribed constant fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. These are elliptic equations for the potentials ϕ_e and ϕ_m, which are solvable with standard numerical methods such as the finite element method, as long as these are implemented with the proper boundary conditions. The potentials depend linearly on $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, which defines linear operators γ_e and γ_m according to the integrals

$$\int_{-\infty}^{\infty}\int_U(\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})(\boldsymbol{E}_0 - \nabla\phi_e)\,\mathrm{d}S\,\mathrm{d}z \stackrel{\text{def}}{=} \gamma_e\boldsymbol{E}_0 \tag{58}$$

$$\int_{-\infty}^{\infty}\int_U(\boldsymbol{\mu}/\mu_0 - \mathbf{I})(\boldsymbol{H}_0 - \nabla\phi_m)\,\mathrm{d}S\,\mathrm{d}z \stackrel{\text{def}}{=} \gamma_m\boldsymbol{H}_0. \tag{59}$$

These matrices are the polarizability matrices in the static limit, used in the preceding section.

The polarizabilities can be defined as the minimum of an energy functional. It is shown in [20] that if $\boldsymbol{\epsilon}(\boldsymbol{x}) \leqslant \boldsymbol{\epsilon}(\boldsymbol{x})'$ for all $\boldsymbol{x}$, the corresponding polarizabilities satisfy $\gamma_{\rm e} \leqslant \gamma_{\rm e}'$. Even though the derivation in [20] is for a single isotropic particle, the arguments are valid for an anisotropic periodic setting as well as is seen in [21], and the corresponding result applies also to $\gamma_{\rm m}$. Thus, the polarizabilities are monotone in the material parameters. In addition, we have the simple estimates [21]

$$\int_{-\infty}^{\infty}\int_U (\mathbf{I} - \boldsymbol{\epsilon}^{-1}\epsilon_0)\,\mathrm{d}S\,\mathrm{d}z \leqslant \gamma_{\rm e} \leqslant \int_{-\infty}^{\infty}\int_U (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})\,\mathrm{d}S\,\mathrm{d}z \tag{60}$$

$$\int_{-\infty}^{\infty}\int_U (\mathbf{I} - \boldsymbol{\mu}^{-1}\mu_0)\,\mathrm{d}S\,\mathrm{d}z \leqslant \gamma_{\rm m} \leqslant \int_{-\infty}^{\infty}\int_U (\boldsymbol{\mu}/\mu_0 - \mathbf{I})\,\mathrm{d}S\,\mathrm{d}z \tag{61}$$

corresponding to the harmonic and arithmetic means of the material parameters. In classical homogenization theory, the corresponding bounds are known as the Wiener bounds [22].

4.2. PEC inclusions

With some small modifications, the above reasoning applies also for metal inclusions in the unit cell. Modeling the metal as a perfect electric conductor (PEC), the equations should then be interpreted as being valid in the domain $U \times \mathbb{R}\backslash\Omega$, where Ω denotes the PEC region and the boundary conditions $\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{E}}^{(0)} = \mathbf{0}$ and $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)}) = 0$ apply on $\partial\Omega$ [23, p 204]. This corresponds to taking the limits $\boldsymbol{\epsilon} \to \infty$ and $\boldsymbol{\mu} \to \mathbf{0}$ in the PEC region.

We identify the $\gamma_{\rm e}$ and $\gamma_{\rm m}$ matrices as giving the total electric and magnetic dipole moment, respectively. Their definitions are then replaced with (using that the surface charge density is $\rho_{\rm S} = \hat{\boldsymbol{n}} \cdot (\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)})$ and the surface current density is $\boldsymbol{J}_{\rm S} = \hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{H}}^{(0)}$)

$$\gamma_{\rm e}\boldsymbol{E}_0 \stackrel{\rm def}{=} \int_{U\times\mathbb{R}\backslash\Omega} (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})\widetilde{\boldsymbol{E}}^{(0)}\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot \frac{\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)}}{\epsilon_0}\,\mathrm{d}S \tag{62}$$

$$\gamma_{\rm m}\boldsymbol{H}_0 \stackrel{\rm def}{=} \int_{U\times\mathbb{R}\backslash\Omega} (\boldsymbol{\mu}/\mu_0 - \mathbf{I})\widetilde{\boldsymbol{H}}^{(0)}\mathrm{d}V + \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{H}}^{(0)})\,\mathrm{d}S. \tag{63}$$

It is shown in [24] that the magnetic polarizability $\gamma_{\rm m}$ for PEC bodies in vacuum is negative. The electric and magnetic polarizabilities are monotone with the volume in the respect that $\gamma_{\rm e} \leqslant \gamma_{\rm e}'$ and $-\gamma_{\rm m} \leqslant -\gamma_{\rm m}'$ if $V \leqslant V'$, where V and V' are the corresponding volumes [25]. In [21] it is shown that these results apply also when the PEC body is surrounded by a fixed anisotropic medium. Furthermore, we have the following estimates for PEC bodies in vacuum [25]:

$$3V' \leqslant \gamma_{\rm e} \leqslant 3V'' \tag{64}$$

$$3V'/2 \leqslant -\gamma_{\rm m} \leqslant 3V''/2, \tag{65}$$

where V' is the volume of the largest sphere contained in the body, and V'' is the volume of the smallest sphere containing the body.

In the following subsection, we show that the last term in (63) is absent if the metal is modeled with a finite conductivity instead of PEC.

4.3. Conducting inclusions

The conductivity case, where $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}' + \boldsymbol{\sigma}/(-\mathrm{i}\omega)$, is fundamentally different since the electric current has a zeroth-order term in the ω expansion due to

$$-\mathrm{i}\omega\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}} = \boldsymbol{\sigma}\widetilde{\boldsymbol{E}} - \mathrm{i}\omega\boldsymbol{\epsilon}'\widetilde{\boldsymbol{E}}. \tag{66}$$

The formal expansions (51) and (52) then imply the following equations for the zeroth-order fields:

$$\nabla \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0} \qquad \nabla \cdot (\boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)}) = 0 \tag{67}$$

$$\nabla \times \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)} \qquad \nabla \cdot (\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)}) = 0. \tag{68}$$

Assuming $\boldsymbol{\sigma} \neq \boldsymbol{0}$ only inside the region Ω implies the boundary condition $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)}) = 0$ at $\partial\Omega$. In simply connected regions Ω there can be no static current, which implies $\boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$. This can be seen in a more formal way by considering the quadratic form (using that $\nabla \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ implies the representation $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}}$ in a simply connected region)

$$\int_\Omega (\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}}) \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}})]\,\mathrm{d}V = \boldsymbol{E}_0 \cdot \int_\Omega \boldsymbol{\sigma}(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}})\,\mathrm{d}V$$
$$+ \int_\Omega \phi_{\mathrm{e}}\nabla \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \phi_{\mathrm{e}})]\,\mathrm{d}V - \oint_{\partial\Omega} \phi_{\mathrm{e}}\hat{\boldsymbol{n}} \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \phi_{\mathrm{e}})]\,\mathrm{d}V. \tag{69}$$

Each of the integrals on the right-hand side is zero: the first because the net static current in a closed region must be zero[1], the second because of the field equation $\nabla \cdot (\boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)}) = 0$ and the third and last due to the boundary condition $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\sigma}\widetilde{\boldsymbol{E}}^{(0)}) = 0$. Since the integrand on the left-hand side is non-negative, it must be zero almost everywhere, proving that the field $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\varphi = \boldsymbol{0}$ in the inclusion geometry Ω. This means the equations for the magnetic field reduce to $\nabla \times \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{0}$ and $\nabla \cdot (\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)}) = 0$, i.e. the metal inclusions do not influence the magnetic field.

To determine $\widetilde{\boldsymbol{E}}^{(0)}$ in regions where $\boldsymbol{\sigma} = \boldsymbol{0}$, we need to consider equations further down the chain,

$$\nabla \times \widetilde{\boldsymbol{E}}^{(1)} = \mathrm{i}\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)} \tag{70}$$

$$\nabla \times \widetilde{\boldsymbol{H}}^{(1)} = -\mathrm{i}\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)} + \boldsymbol{\sigma}\widetilde{\boldsymbol{E}}_1. \tag{71}$$

Taking the divergence of the last equation, it is seen that in regions where $\boldsymbol{\sigma} = \boldsymbol{0}$, i.e. outside Ω, we necessarily have $\nabla \cdot (\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)}) = 0$. Since $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ inside Ω and the tangential electric field must be continuous, this implies the standard boundary condition $\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ on $\partial\Omega$.

To summarize, if the metallic inclusions are modeled with a finite conductivity, the electric polarizability should be calculated just as in the PEC case, but the magnetic polarizability is only due to variations in $\boldsymbol{\mu}$. The physical difference between the two models is that in the PEC case, the low frequency limit is taken such that an infinitesimal skin depth is maintained in the metallic particle, whereas in the conductivity case the limit is taken so that the skin depth is much greater than the particle. From (46), we see that the first-order transmission coefficient is the sum of electric and magnetic polarizabilities, and from (47) the first-order reflection coefficient is the difference. Since the magnetic polarizability is negative for PEC bodies, we conclude that the difference between the two models is that the first-order transmission is smaller for the PEC model than for the conductivity model, whereas the first-order reflection is larger for the PEC model than for the conductivity model.

[1] Mathematically, this can be shown by considering the integral $\int_\Omega \nabla \cdot (x\vec{J})\,\mathrm{d}V$. Using the divergence theorem, we have $\int_\Omega \nabla\cdot(x\vec{J})\,\mathrm{d}V = \oint_{\partial\Omega} \hat{\vec{n}}\cdot(x\vec{J})\,\mathrm{d}S = 0$ due to the boundary condition $\hat{\vec{n}}\cdot\vec{J} = 0$. Using that $\nabla\cdot(\varphi\vec{J}) = \nabla\varphi\cdot\vec{J} + \varphi\nabla\cdot\vec{J}$ for any φ, the integral must also be $\int_\Omega J_x\,\mathrm{d}V$. Thus, all components of the current integrate to zero, and we have $\int_\Omega \vec{J}\,\mathrm{d}V = \boldsymbol{0}$ for closed regions.

5. Conclusions

In this paper, we have derived the first-order asymptotic behavior for the low-frequency reflection and transmission coefficients of a low-pass periodic structure. The structure can be anisotropic in both electric and magnetic properties, and the angle of incidence as well as the polarization of the incoming wave is arbitrary. It is found that the low-frequency behavior is proportional to the static electric and magnetic polarizability per unit area of the periodic structure. The transmission coefficient is associated with the sum of the polarizabilities, and the reflection coefficient with the difference.

The polarizabilities can be considered as minima of energy functionals, which provide simple estimates in terms of easily calculated quantities associated with the harmonic and arithmetic mean of the material parameters. When modeling the metal inclusions with a finite conductivity instead of as PEC, the electric polarizability is unchanged, i.e. a specific dipole moment can be identified for the metal body, whereas the magnetic polarizability only depends on variations in permeability, with no specific contribution from the metal body.

The strong variational results for the static problem makes it possible to compute estimates even for very complicated structures, and use these as bounds. It is anticipated that the results of this paper can be used to formulate physical limitations on the amount of electromagnetic interaction that is available from a structure with a finite thickness.

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